GOVERNMENT OF ANDHRA PRADESH
COMMISSIONERATE OF COLLEGIATE EDUCATION

PROPERTIES OF FOURIER TRANSFORMS

MATHEMATICS

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CONTENTS

Learning Outcomes

- Able to learn how to find Fourier transforms and inverse Fourier transforms of some standard functions.
- Able to learn how to find finite Fourier sine and cosine transforms and apply it in solving boundary value problems.
- Comprehend the properties of Fourier transforms and solve problems related to infinite Fourier transforms.
- Classify and solve partial differential equations.
- The Fourier transform can be used to interpolate functions and to smooth signals.
- Fourier transforms can be used to determine the constituent pitches in a musical waveform.

Definition - Fourier Transform :

Let $f(x)$ be a function defined on $(-\infty, \infty)$ and be piecewise continuous in each finite partial interval and absolutely integrable in $(-\infty, \infty)$.

Then the Fourier Transform of $f(x)$ is defined as

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx$$

Hence,

$$\mathrm{F}\{f(x)\} = \bar{f}\,(\mathrm{p}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx$$

➢ Linearity property of Fourier Transform.

If $\bar{f}$(p) and $\bar{g}$(p) are Fourier Transform of f(x) and g(x) respectively then

$$F\{af(x)+bg(x)\}=a\,\bar{f}(p)+h\,\bar{g}(p)$$

Where a and b are constants.

Proof : we have

$$F\{f(x)=\bar{f}(p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(x)\,dx$$

$$F\{g(x)\}=\bar{g}(p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} g(x)dx$$

$$F\{af(X)+bg(x)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx}\,(af(x)+bg(x)\}\,dx$$

$$=\frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(x)\,dx+\frac{b}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} g(x)dx$$

$$=a\,\bar{f}(p)+b\,\bar{g}(p)$$

Change of Scale Property :

Theorem 1:

If $\bar{f}(p)$ is the complex Fourier Transform of f(x) then

$$F\{f(ax)\} = \frac{1}{a}\bar{f}\left(\frac{p}{a}\right), a > 0$$

Proof:

If $\bar{f}(p)$ is the complex Fourier Transform of f(x) then

$$\bar{f}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(x)\, dx$$

$$\therefore F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(x)\, dx \quad \text{--------} (1)$$

To prove that,

$$F\{f(ax)\} = \frac{1}{a}\bar{f}\left(\frac{p}{a}\right)$$

From equation (1),

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(ax)\, dx$$

Put ax = t so that adx = dt

⇨ dx = dt/a

Limits :

If $x \to -\infty$ then $t \to -\infty$ and

If $x \to \infty$ then $t \to \infty$

Hence,

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ip(\frac{t}{a})} f(t)\ \frac{dt}{a}$$

$$= \frac{1}{a}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\,(\frac{p}{a})t} f(t)\, dt$$

$$= \frac{1}{a}\ \bar{f}(\frac{p}{a})$$

$$\boldsymbol{F\{f(ax)\} = \frac{1}{a}\bar{f}(\frac{p}{a})}$$

Shifting Property :

Theorem :

Statement : If $\bar{f}(p)$ is the complex Fourier transform of $f(x)$, then the complex Fourier transform of

$f(x-a)$ is $e^{ipa}\,\bar{f}(p)$

i.e., $F\{f(x-a)\} = e^{ipa}.\ \bar{f}(p)$

Proof :

By the definition of Fourier transform of $f(x)$, we have

$$F\{f(x)\} = \bar{f}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx \quad \text{------------------------------>} \ (1)$$

$$\therefore F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(x-a)\, dx$$

When $x \to -\infty, t \to -\infty$ and if $x \to +\infty, t \to +\infty$.

i.e., $F\{ f(x-a) \} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ip(a+t)} f(t)\, dt$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipa}\, e^{ipt} f(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} e^{ipa} \int_{-\infty}^{\infty} e^{ipt}\, f(t)\, dt$$

$$= e^{ipa}\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipt}\, f(t)\, dt \}$$

$$= e^{ipa}\, \bar{f}(p) \qquad \text{[from (1)]}$$

$\therefore F\{ f(x-a) \} = e^{ipa}\, \bar{f}(p)$

Hence the theorem is proved.

References:

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3. Bracewell, The Fourier Transform and its applications (3 rd ed.), Boston: McGraw – Hill.

4. James, J.F. A Student's Guide to Fourier Transform, Cambridge University Press.

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